Title	RANDOM MATRIX FORMULATIONS TOWARD QUANTUM INFORMATION THEORY
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Citation	数理解析研究所講究録 (1994), 859: 71-90
Issue Date	1994-03
URL	http://hdl.handle.net/2433/83812
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
859巻 1994年 71-90

RANDOM MATRIX FORMULATIONS TOWARD QUANTUM INFORMATION THEORY

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A brief account of the development of energy level statistics is given: (A) the original random-matrix framework, (B) Berry's reformulation of the results of (A) by means of the periodic orbit sums in the trace formula of Gutzwiller, and (C) dynamical approaches to the matrix ensembles in (A). The white noise characteristic of the periodic orbit sum for a level density found by Berry in (B) is discussed to provide a supporting basis of a stochastic formulation as extended from the approach in (C). This revises Dyson's Brownian motion model properly. Finally, Yukawa's distribution obtained in (C) is characterized by the maximum information principle.

1. INTRODUCTION

Studies of energy-level fluctuations for quantum systems can be classified in their historical development as three phases; the first development around 1960 to form the old random-matrix theory, the second around 1980 in connection with chaos in classical mechanics, and the third since 1985 in terms of level dynamics which may be called dynamical random-matrix theory. We discuss some of the above features with concrete example of our computer results on a circular and a stadium billiard. It will be seen that these three phases are intimately related to one another, and the last phase (C) seems to yield a possibility of recapitulating the result as information theory.

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presented at RIMS meeting (March 1991) Quantum Information Theory

2. THREE ASPECTS OF THE DEVELOPMENT OF LEVEL STATISTICS

(A) the old random matrix theory

Main contributors to this theory are well-knon: Wigner, Porter, Mehta and Dyson, and a comprehensive review for their studies is given in Porter's book [1] (see also Mehta [2]). The theme of this theory is first to construct the joint distribution function for N eigenvalues $\{x_n\}$ of an N×N Hamiltonian (truncated) matrix and on this basis to deduce various ingredients for level statistics: important ones are the two-level cluster function $Y(x_1, x_2)$ and the nearest-neighbor(NN) spacing distribution function P(S) with $S=x_{n+1}-x_n$ $(\cdots x_n \leq x_{n+i} \cdots)$.

The N-level joint distribution function established is written as

$$P_\nu(\{x_n\}) = C_{N\nu} \exp\left(-\frac{1}{4a^2}\sum_{n=1}^{N} x_n^2\right) \prod_{m<n} |x_m - x_n|^\nu \qquad (1)$$

for the ensemble of N×N real symmetric matrices with $\nu=1$, complex hermitian ones with $\nu=2$, and, in case N=even, for that of symplectic-invariant hermitian matrices with $\nu=4$: these are the basic classification of Hamiltonians in terms of the time-reversal i.e.

$$\left.\begin{array}{ll} \nu=1 & \text{time-reversal invariant with even number of spins} \\ 2 & \text{noninvariant} \\ 4 & \text{invariant with odd number of spins,} \end{array}\right\} \qquad (1a)$$

for which the ensemble (1) is designated as Gaussian Orthogonal, Gaussian Unitary and Gaussian Symplectic ensemble (GOE,GUE,GSE), respectively.

The Gaussian-ensemble distribution (1) is the consequence of the three requirements that

i) every matrix elements of a matrix H belonging to the ensemble specified are independent random variables,

ii) the function $P_\nu(\{x_n\})$ is invariant by all the symmetry transformations proper to each ensemble (O(N) for GOE, U(N) for GUE and S(N/2) for GSE),

iii) the center of the gravity of $\{x_n\}$ vanishes,i.e. TrH=0.

The last requirement causes an inconvenience that the resulting distribution function does not satisfy the translational invariance

$$P(\{x_n+a\}) = P(\{x_n\}) \qquad (2)$$

and hence the same inconvenience for all the distribution functions deduced by partial integrations of $P(\{x_n\})$ over several variables (e.g. $\int P_{\nu=1}(\{x_n\})dx_2 \cdots dx_N$ gives Wigner's semi-circle law [1], and $\int P_\nu(\{x_n\})dx_3 \cdots dx_N$ gives $Y(x_1, x_2)$ which is not a function of the single variable $x_1 - x_2$, etc.), thus giving rise to considerable technical difficulties for further development. Dyson's distribution functions for circular ensembles (eigenvalue distributions for unitary matrices) which go parallel with the three cases $\nu=1$, 2 and 4 in (1a) remove the difficulty, but still involve technical complexity for practical applications. The two-level cluster function $Y(x)(=Y(|x_1 - x_2|))$ obtained from the nonuniform one by using some technics involving a 'thermodynamic limiting process' $N \to \infty$, has achieved a satisfactory description of the correlation properties of the levels subject to the Gaussian prediction (1) (the so-called Δ-statistics: see below, also Bohigas and Giannoni [3]).

(B) Berry's context of introducing classical periodic orbits

The expectation that the results of the old random-matrix theory could be reformulated in the framework of the semiclassical mechanics (quantizing the periodic orbits of a Hamiltonian mechanics) was stated by Berry and Tabor [4] early in 1977. In this paper, it was verified, both logically and in numerical computations, that an integrable classical system with f degree of freedom having f constants of motion must be subject, with some reservations, to the Poisson statistics where the NN spacing distribution function is a simple exponential: $P(s)= e^{-s}$. This work gave a considerable impetus to the later development, and after ten years the expectation was realized as an accepted form in Table 1, although the actual implication of this requires more refined details (such as nongeneric integrable systems of harmonic oscillators, etc.).

Table 1. Accepted understanding of the periodic-orbit sum theory

classical Hamilton system with f degrees of freedom ($f \geq 2$)	level statistics for the corresponding quantum system
complete integrable system	Poissonian (level clustering)
non-integrable system	Gaussian* { GOE $\nu=1$ in (1); GUE $=2$; GSE $=4$ }

* with the factor $\prod_{m<n} |x_m - x_n|^\nu$, which causes the level repulsion.

A comprehensive account of Table 1 can be seen in Berry's semiclassical theory of spectral rigidity [5] in 1985 (see also [6]): He shows a periodic-orbit sum version of the statistical quantity $\Delta(L)$ of Dyson and Mehta in (A) defined as the minimum averaged variance of the level numbers in the interval L of the unfolded energy scale [3], i.e.

$$\Delta(L;\alpha) = \min_{A,B} \frac{1}{L} \int_{-L/2+\alpha}^{L/2+\alpha} \overline{(N(x)-Ax-B)^2}\,dx \tag{3}$$

$$= \frac{1}{2L^2} \int_{-L/2+\alpha}^{L/2+\alpha} dy \int_{-L/2+\alpha}^{L/2+\alpha} dx \left(1+\frac{12xy}{L^2}\right) \overline{(N(x)-N(y))^2}. \tag{3a}$$

In these expressions, x, y, stand for the energy variables subject to the unfold mapping:

$$E \to N_{av}(E) \equiv x,$$

when the level numbers $\leq E$, N(E), is decomposed into parts of average and fluctuations such that

$$N(E) = N_{av}(E) + N_{fl}(E), \tag{4}$$

the average being specified by $\langle\cdot\rangle = L^{-1}\int_{-L/2}^{L/2} dE$: thus $N_{av}(E)$ is given by the phase volume E (the so called Thomas-Fermi value). The other average involved in (3) indicated by $\overline{\ }$ is the average over the fluctuations of level numbers (in a specified interval) so that

$$\overline{(N(x)-N(y))^2} = \overline{(x-y-N_{fl}(x)+N_{fl}(y))^2} = (x-y)^2 + \Sigma^2(|x-y|), \tag{5}$$

with $\Sigma^2(S)$ being the number variance in the interval [0, S] which is related to the 2-level cluster function Y(S) as

$$\Sigma^2(S) = S - 2\int_0^S (S-r)Y(r)\,dr. \tag{6}$$

Expressions (5) and (6) are used in (3a) to compute $\Delta(L)$ in terms of the cluster function Y (for this version of computing $\Delta(L)$, see [3]).

Berry [5] gave another expression for computing $\Delta(L)$, which reads

$$\Delta(L) = \frac{1}{k^{K-1}} \int_0^\infty \frac{dT}{T^2} \phi(T) G\left(\frac{LT}{2k\langle\rho\rangle}\right) \qquad (K=f: \text{integrable, or } 1: \text{chaotic}) \tag{7}$$

with the orbit selecting function $G(y) = 1 - \sin^2 y/y^2 - 3\left(\frac{d}{dy}\sin y/y\right)^2$ (7a)

(independent of the object system),

and the spectral form factor (depending on the system)

$$\phi(T) = \left\langle \sum_i \sum_j^{+} A_i A_j \cos\{(S_i - S_j)/\hbar\}\, \delta\left(T - \frac{T_i+T_j}{2}\right) \right\rangle \qquad (7b)$$

(i,j denote periodic orbits with action S_i, S_j and period T_i, T_j for a given E). Expressions (7), (7a, b) stem from the local-energy expansion of the action funnction

$$S_j(E+\varepsilon) = S_j(E) + T_j(E)\,\varepsilon \qquad (7c)$$

in the periodic-orbit sum

$$\rho_{os}(E)\left(= \frac{d}{dE} N_{fl}(E)\right) = \frac{1}{\hbar^{(K+1)/2}} \sum_j A_j\, e^{iS_j/\hbar} \qquad (8)$$

which may be understood as a convention for the unfolding of energy scale.

One of the most important findings by Berry [5] is the asymptotic sum rule satisfied by the form factor $\phi(T)$:

$$\phi(T) = \langle \rho \rangle \frac{\hbar^K}{2\pi} \qquad \text{for } T \gg \langle \rho \rangle \hbar \qquad (9)$$

so that $\lim_{T\to\infty} \phi(T) = \text{constant} < \infty$ valid for any (bound) system. For integrable systems (K=f) the equality of (9) holds much wider range of T:

$T \geq T_{min}$(the minimum of all the possible periods of periodic orbits). (9a)

The form factor $\phi(T)$ is the Fourier transform of the autocorrelation function of the level-density function on the local-energy scale i.e.

$$\phi(T) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \langle \rho_{os}(\lambda+\varepsilon)\, \rho_{os}(\lambda) \rangle\, e^{i\varepsilon T/\hbar} d\varepsilon \qquad (10)$$

(the power spectrum of the oscillating part level density $\rho_{os}(\varepsilon)$),

where the stationary property $\langle \rho_{os}(\lambda+\varepsilon)\, \rho_{os}(\lambda) \rangle$=independent of λ holds by virtue of the linearizing approximation (7c): Thus, the constancy of the power spectrum of $\rho_{os}(\varepsilon)$ implies that it possesses the characteristic of white noise.

Indeed, the essence of Berry and Tabor's proof of the Poisson statistics ($P(S)=e^{-S}$) for integrable systems consists in establishing the general relation between the autocorrelation function $\langle \rho_{os}(S)\, \rho_{os}(0) \rangle$ and the cluster function Y(S) as follows (see Appendix B in [4]):

$\rho(S) = 1 + \rho_{os}(S)$ (normalization $\langle \rho \rangle = 1$ by the unfold map)

$$\langle \rho_{os}(s)\rho_{os}(0)\rangle = \delta(s) - Y(s)$$
$$= \delta(s) - 1 + \sum_{k=0}^{\infty} P_k(s) \qquad (11)$$

where $P_k(S)$ stands for the conditional probability that k-levels are contained in the interval [0, S] when the end points 0 and S are both filled, and hence $\sum P_k(S)$ for the probability that 0 and S are filled irrespective of levels present or absent inside of the interval. By explicit evaluation of the power spectrum for integrable systems, they show

$$\langle \rho_{os}(s)\rho_{os}(0)\rangle = \delta(s). \qquad (12)$$

Thus, from (11) and Porter's relation $P(S)=g(S)e^{-\int_0^S g(r)dr}$ ([1], under the assumption of independence between non-overlapping intervals; $g(r)=\sum P_k(r)$) assures the validity of $P(S)=e^{-S}$.

Actually, the white noise characteristic (12) must be modified because of the presence of the minimum period of the periodic orbits (9a): This causes the saturation of $\Delta(L)$ beyond the value $L_{max}=\hbar/T_{min}$ [5].

We present in Figs. 1a~d our numerical examples of the level statistics for billiards (integrable one for a circle and nonintegrable one for a stadium).*

(C) dynamical random-matrix theory

The original idea by Porter [1] to deduce the joint distribution function (1) for the Gaussian ensemble was to introduce a Riemann metric into the matrix space specified: we show this by way of example for the case of real symmetric N×N matrices (GOE with $\nu=1$).

$$(dS)^2 = Tr(dH)^2 = \sum_n (dH_{nn})^2 + 2\sum_{m<n} (dH_{mn})^2 \qquad (13)$$

for the N diagonal elements H_{nn}, and

$$Z=\frac{1}{2}N(N-1) \quad \text{off-diagonal elements } H_{mn}(m<n). \qquad (13a)$$

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* Results based on the work of master thesis by T. Takami (1991). The numerical procedure is given in the Appendix.

Fig. 1

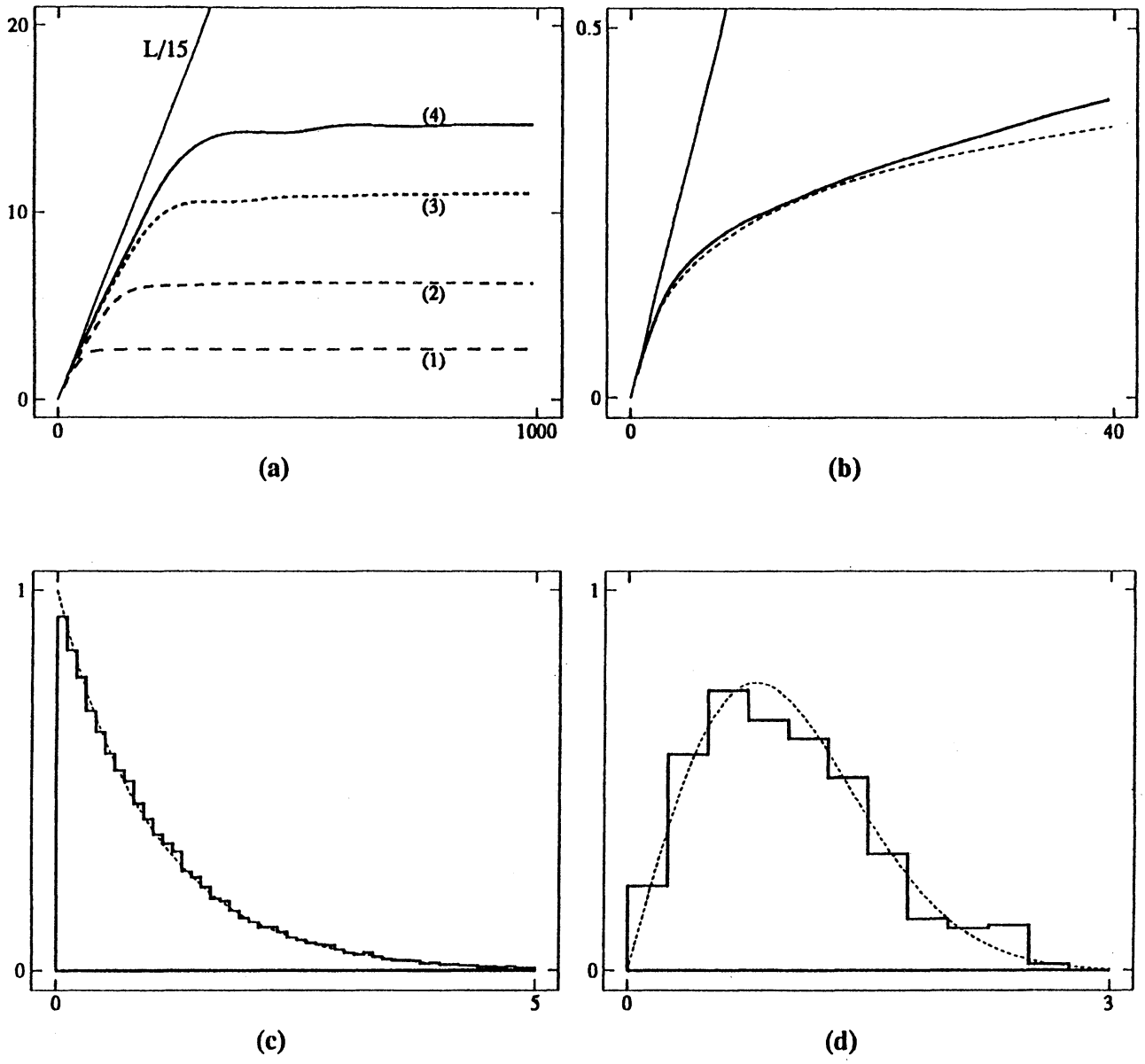

Fig.1

(a) Statistical quantity $\bar{\Delta}(L)$ for a circlular billiard. Eigenvalues (determined from zeros of the Bessel function J_m) in the intervals (on the unfolded energy scale)
(1) 0–2000, (2) 0–10000, (3) 10000–20000, (4) 20000–30000
are used to compute $\bar{\Delta}(L)$. Each curve exhibits a behavior of saturation due to the finiteness of the number of levels. This corresponds to the finiteness of the Planck constant $\hbar$, and is expected to vanish in the limit $\hbar \to 0$ corresponding to the ideal Poisson statistics. The saturation was demonstrated for the first time by Casati *et al.* for rectangular billiards[19].

(b) $\bar{\Delta}(L)$ for a studium billiard, which is compared with the theoretical curve for GOE(dotted).

(c) NN spacing distribution for a circular billiard, which is compared with $\exp(-S)$.

(d) NN spacing distribution for a stadium billiard, which is compared with Wigner's distribution.

In the representation that H is diagonal i.e.

$$O^{-1}HO = diag(x_1, x_2, \cdots x_n) \equiv H_D \tag{13b}$$

the metric is rewritten as

$$\begin{aligned} ds^2 &= Tr(dH_D)^2 + Tr([H_D, \Omega][H_D, \Omega]) \\ &= \sum_n (dx_n)^2 + 2\sum_{m<n} (x_m - x_n)^2 \Omega_{mn}^2 \end{aligned} \tag{14}$$

where Ω is the matrix one-form

$$\Omega = (\Omega_{mn}) = O^{-1}dO \qquad (O^{-1} = O^T) \tag{14a}$$

which is not closed (i.e. $d\Omega \neq 0$) but can be regarded as constituting, together with N-infinitesimals ($dx_1, dx_2, \cdots dx_N$), a proper transformation

$$(dH_{mn})_{m \leq n} \rightarrow (dx_1, dx_2, \cdots dx_N ; \Omega_{mn}\ (m<n)) \tag{15}$$

to satisfy the relation of measures in the both representations

$$meas(dH) \equiv \prod_{m \leq n} dH_{mn} = \prod_n dx_n \prod_{m<n} |x_m - x_n| \Omega_{mn} \tag{16}$$

where the 'level repulsion factor', $\prod_{m<n} |x_m - x_n|$, arises from the metric (14) i.e. as the square-root of the metric coefficient determinant

$$\sqrt{g} = (\det(g_{mn}))^{1/2} = \prod_{m<n} |x_m - x_n| \tag{16a}$$

(the factor 2^2 which arises from (14) is omitted).

Note that the GOE distribution function (1) with $\gamma = 1$ can be regarded as the expression after integrated out over the partial measure $\prod_{m<n} \Omega_{mn}$ for the space of N-dimensional rotations which should yield a finite value (see RIMS preprint by Hasegawa and Ojima [7]).

The two representations of the Riemann metric (13) and (14) should yield a system of classical mechanics through the kinetic energy

$$T = \frac{1}{2}\left(\frac{ds}{d\lambda}\right)^2 = \frac{1}{2} Tr(dH/d\lambda)^2 \tag{17}$$

$$= \frac{1}{2} \mathrm{Tr}(dH_D/d\lambda)^2 + \frac{1}{2} \mathrm{Tr}(\Omega_\lambda \mathcal{A} \Omega_\lambda), \tag{17a}$$

where the time variable is denoted by λ and in the representation (17a),

$$\Omega_\lambda = (\frac{d}{d\lambda} O^{-1}) dO, \quad \text{and from (14)} \tag{18}$$

$$\mathcal{A}\Omega_\lambda = -[H_D, [H_D, \Omega_\lambda]]. \tag{18a}$$

The mechanics is Lagrangian with the Lagrangian $\mathcal{L}(\dot{q}, \dot{q}) = T$ to yield a free motion; in the first representation, a Newtonian mechanics governed by $d^2H/d\lambda^2 = 0$ whose solution must be given in terms of two (real symmetric) matrices H° and V as

$$H = H^\circ + \lambda V. \tag{19}$$

Then, in the second representation the motion should yield the N eigenvalues of $H^\circ + \lambda V$ for the first subset $\{x_n(\lambda)\}$ and a trajectory for the second subset which must be a geodesic in the Z-dimensional rotation space O(N); an analogy in the N-dimensional Euclidean space to Euler's motion of a rigid body in 3-dimension.

The free motion (19) can be regarded as a change of the Hamiltonian H by a perturbation V in the ensemble of all the real symmetric N×N matrices: this consideration would yield a new framework of finding the distributions for H via the standard statistical mechanical treatment, the idea first proposed by Yukawa [8] and followed by several authors [9]~[11]. The framework can be constructed naturally from the Lagrangian formulation to the Hamiltonian one where the canonical variables should be set for $\frac{1}{2}N(N+1)$ coordinates X_{mn} chosen by H_{mn} and their conjugate momenta P_{mn} by $2V_{mn}$ in the first representation. To go over to the second representation, one should investigate the transformation of the variables in the phase space

$$(X_{mn}, P_{mn}) \longrightarrow (x_n, p_n) \times (\Omega_{mn}, M_{mn}) \tag{20}$$

such that

$$\sum_{m \le n} P_{mn}\, dX_{mn} = \sum_n p_n\, dx_n + \sum_{m<n} M_{mn} \Omega_{mn} \tag{20a}$$

This is not a canonical tranformation: the fundamental Poisson bracket relations $\{X_{mn}, P_{st}\} = \delta_{ms}\delta_{nt}$ valid in the left-hand side cannot be preserved in the right-hand side. For the four sets of the variables in the right-hand side of (20), however, the necessary Poisson bracket relations can be deduced by

taking the exterier derivative [12] of the one-form present in the representation (20a) and by noting the so-called Maurer-Cartan equation [13]:

$$d\Omega = -\Omega \wedge \Omega \qquad \text{(matrix two-form)}. \tag{21}$$

A more detailed discussion of deducing the brackets is given in [7], where one finds

$$\begin{aligned} &\text{(a)} \quad \left\{ \begin{pmatrix} x_\ell \\ p_\ell \end{pmatrix}, \begin{pmatrix} M_{mn} \\ \Omega_{\lambda\, mn} \end{pmatrix} \right\} = 0 \\ &\text{(b)} \quad \{ x_m , p_n \} = \delta_{mn} \\ &\text{(c)} \quad \{ M_{mn} , M_{st} \} = \delta_{nt} M_{sn} + \delta_{ns} M_{tm} + \delta_{ms} M_{nt} + \delta_{mt} M_{ms} \end{aligned} \tag{22}$$

(the angular mnomentum relations in the N-dimensional space).

This is in agreement with the relations set up without reasoning by Gaspard et al [11] (the factor 1/2 present in their relations for $\{M_{mn}, M_{st}\}$ is due to the difference in defining M from the present one).

The Lagrangian mechanics of the free matrix motion (19) can now be put in the Hamiltonian form with Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{m \le n} P_{mn}^2 = \frac{1}{2} \sum_n p_n^2 + \frac{1}{4} \sum_{m<n} \frac{M_{mn}^2}{(x_m - x_n)^2} , \tag{23}$$

and Yukawa [8] put this form together with another constant of motion $Q = -\mathrm{Tr}M^2$ into the partition function (partial integrations of the Boltzmann factor $e^{-\beta\mathcal{H} - \gamma Q}$ over the irrelevant p and M-variables) to get a new distribution function for $\{x_n\}$.

3. THE TWO TEMPERATURE DISTRIBUTION FUNCTION PROPOSED BY YUKAWA

One of the most fruitful results of the dynamical random-matrix theory is the distribution function obtained by Yukawa [8] to replace the Gaussian-ensemble one (1). (Another important study that has been made recently [11] i.e. the curvature distribution shall be in a separate discussion [22].)
Yukawa's distribution function reads

$$P_\nu(\{x_n\} : \gamma) = C_{N\nu\gamma} \prod_{m<n} \frac{|x_m - x_n|^\nu}{[1 + \gamma (x_m - x_n)^2]^{\nu/2}} \qquad \gamma > 0. \tag{24}$$

A summary of this function (derivation, its advantage over (1) and a practical use) is given below.

1) This function is deduced from the partial integrations (what is called 'coarse-graining' in statistical mechanics) of the Boltzmann factor of the level-dynamical system whose Hamiltonian is given by (23)* over the phase space variables other than $\{x_n\}$:

$$P_\nu(\{x_n\};\gamma) = \int dp d\Omega\, dM \frac{1}{Z_\gamma} e^{-\mathcal{H}-\gamma Q} \qquad (Z_\gamma = \int d\Gamma e^{-\mathcal{H}-\gamma Q}) \tag{25}$$

$$\mathcal{H} = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \frac{1}{4}\sum_{m<n}^{Z} \frac{|M_{mn}|^2}{(x_m - x_n)^2} \tag{25a}$$

$$Q = \sum_{m<n}^{Z} |M_{mn}|^2. \tag{25b}$$

Note that the volume element of the phase space of the system yields two ways of the representation associated with the transformation (18), although it is noncanonical, given by

$$d\Gamma = \prod_{m\le n} dX_{mn} dP_{mn} = \prod_n dx_n\, dp_n \prod_{m<n} \Omega_{mn} dM_{mn}. \tag{25c}$$

2) It does not satisfy the three requirements for the Gaussian ensembles, i) ii), that fix the expression (1). Instead, it satisfies the translational invariance (2): Therefore,

$$P_\nu(\{x_n\};\gamma)dx_2 \cdots dx_N = \text{const.} \qquad (\text{independent of } x_1) \tag{26a}$$

$$P_\nu(\{x_n\};\gamma)dx_3 \cdots dx_N = R(x_1 - x_2), \tag{26b}$$

and, by taking the constant value of (26a) as unity, the uniformity of the cluster function holds automatically, thus (see [3])

$$Y(x_1, x_2) = Y(x_1 - x_2) = 1 - R(x_1 - x_2). \tag{26c}$$

The translational invariance is expected to assure straightforwardly Berry's relation (11).

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* M_{mn}'s are complex ($\nu=2$) or quarternion ($\nu=4$) for which $|M_{mn}|^2$ stands for the sum of ν square terms [11].

3) For $\gamma=0$, it reduces to the level repulsion facter (the expression (1) without the Gaussian exponent), and for $\gamma\to+\infty$ to a constant value almost everywhere in the x-space: this latter situation represents the Poisson statistics (no correlations between different levels) as noted by Yukawa [8]. He also noted that, instead of just setting $\gamma=0$, the process of two limits

$$\gamma\to 0, \quad N\to\infty \text{ with } \gamma N = \text{const } (=1/2\nu a^2) \tag{27}$$

yields a distribution consistent with (1), if the third requirement of vanishing center-of-gravity, $\Sigma x_n=0$, is further imposed: the above limiting procedure is analogous to that in the central limit theorem, thus the distribution (1) can be looked upon as the asymptote of the Yukawa distribution (24) (together with the condition $\Sigma\, x_n=0$) for $N\gg 1$.

4) The actual computation of the cluster function (26c) for arbitrary positive values of γ is difficult. At present, only the approximate form of $Y(x;\gamma)$ with $\gamma\gg 1$ (i.e. for those systems in near integrable situation) is available: this is given by

$$Y(x;\gamma) = 1 - \frac{|x|^{\nu}}{[\gamma^{-1}+x^2]^{\nu/2}} \qquad \nu = \begin{matrix}1 & \text{orthogonal}\\ 2 & \text{unitary}\end{matrix} \tag{28}$$

in analogy to the second virial coefficient of the virial expansion for incomplete classical gases. (See [14])

5) An important remark about the derivation of Yukawa s distribution function (24), namely the coarse-graining integration (25), is the ergodic property of the underlying level dynamics: The canonical distribution $e^{-\mathcal{H}-\gamma\mathcal{Q}}$ with two inverse-temperature parameters (one is unity, the other is γ) looks as if $\mathcal{H}$ and $\mathcal{Q}$ were the only two constants of motion. But this is not the case, becbecause, as Yukawa [8] noted, the level dynamics formulated in (C) is completely integrable. Yukawa and Ishikawa [9] investigated the reason of destruction of most of the constants of motion other than $\mathcal{H}$ and $\mathcal{Q}$ by the inference that confinement of the level motions in a finite range would make the dynamics ergodic. We will try a formulation of statistical mechanics of open systems, where in the representation of the system $\prod(x_n,p_n)\otimes\prod(\mathcal{L}_{mn},M_{mn})$ the latter subsystem is regarded as a kind of heat reservoir.

4. STATISTICAL MECHANICS OF LEVELS AS AN OPEN DYNAMICAL SYSTEM

4.1 the Poisson white noise $W_P(t)$ and its simulation

$$\langle W_P(t) \rangle = 1, \quad \langle [W_P(t_1)-1][W_P(t_2)-1] \rangle = \delta(t_1-t_2) \tag{29}$$

Or, in terms of the integrated expression $\int_0^t (W_P(\tau)-1)\,d\tau \equiv D(t)$.

$$\left.\begin{aligned} &\langle D(t) \rangle = 0, \quad \langle [D(t_1)-D(t_2)]^2 \rangle = |t_1-t_2| \\ &\langle [D(t_1)-D(t_2)][D(t_3)-D(t_4)] \rangle = 0 \ \text{if} \ [t_1,t_2] \cap [t_3,t_4] = \phi \end{aligned}\right\} \tag{30}$$

Thus, the level density on the unfolded energy scale (denoted by t here) $\rho(t) = 1 + \rho_{os}(t)$ for Berry's generic integrable systems satisfying the δ-correlation (12) meets the condition (29) (or, the level number $N(t) = t + N_{fl}(t)$ meets the condition (30)).

More in general, the expression N(t) and that $\rho(t)$ for a (random) quantum system as a sum of step (or δ-) functions satisfying $\langle N(t) \rangle = t$ ($\langle \rho(t) \rangle = 1$) defines a stochastic process with a set of random jump points called Poisson random points. An inspection of Berry's proof [5] of the asymptotic sum rule (9) for the spectral form factor (the power spectrum) assures the validity of the expression (11) with $|Y(t)| < \infty$ everywhere. It implies that, under the assumption of non-overlapping of any pair of different random points, the δ-correlation holds for $\langle \rho_{os}(t)\,\rho_{os}(0) \rangle$ near the origin (i.e. the point where a level is present).

Now, in order to be able to proceed further presently, we introduce a hypothesis that the white noise (29) could be simulated well by another familiar white noise, namely the Gaussian white noise, or for the integrated object D(t) by the Brownian motion. At present, we do not know how serious this approximation would be, which we hope to study: See Fig.2 of the actual motion of $N_{fl}(x)$ produced by our computation on the billiard systems.

4.2 a Brownian motion model in the dynamical approach

The idea goes back to Dyson's work [15] in the old random matrix theory: He imagined that the 'level repulsion factor' $\prod |x_m - x_n|^\nu$ in the distribution function (1) is a consequence of zig-zag motions of each level in a potential field produced by perturbation change of the assumed Hamiltonian matrix, where the perturbing matrix elements are so random that it can be regarded as white noise. In spite of the attractiveness of the idea, however, Dyson's

Fig.2

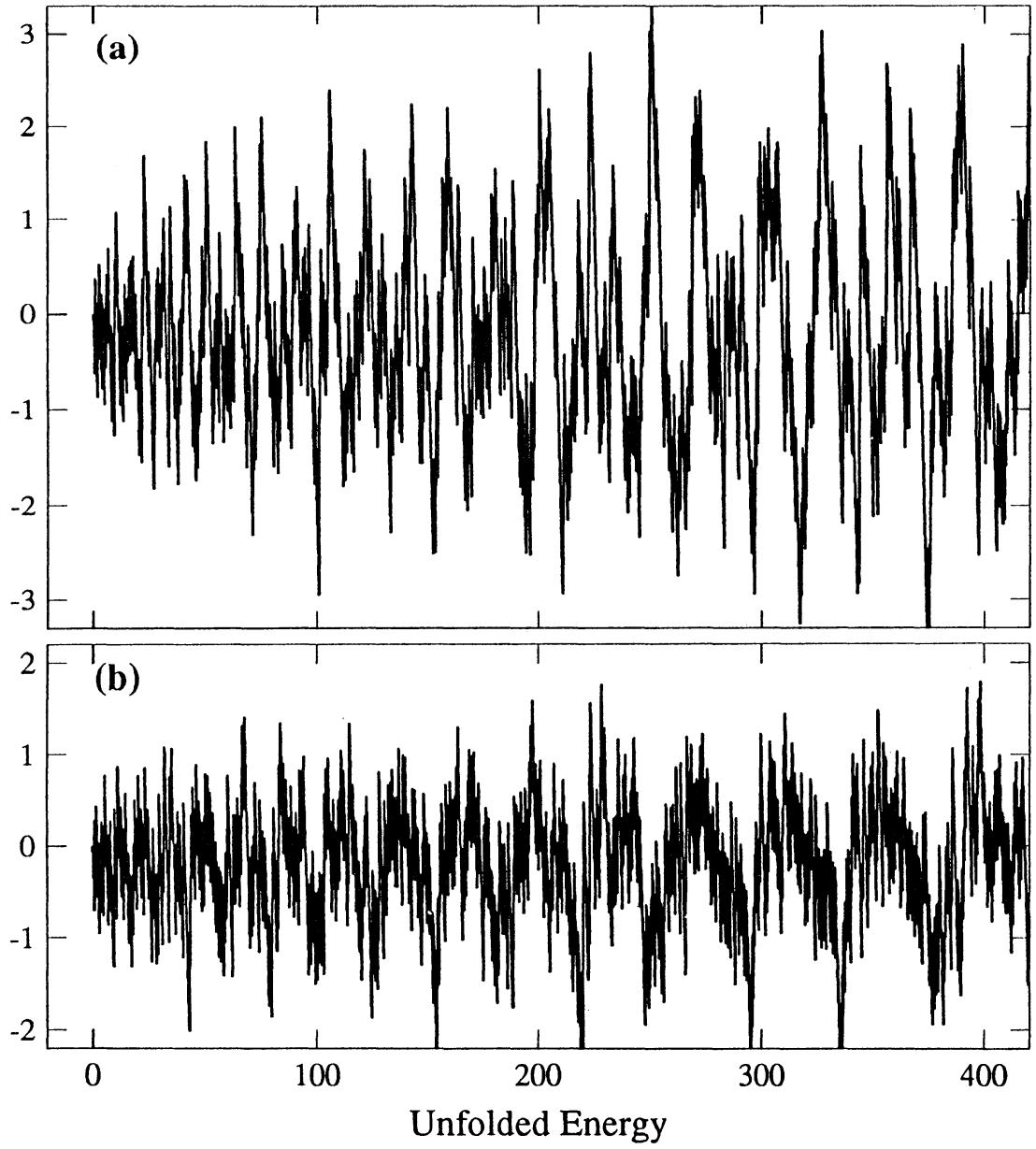

Fig.2 Fluctuation of eigenvalues. The horizontal axis is the unfolded energy and the vertical axis is the difference from the Weyl's formula.

a) Integrable case (circular billiard).

b) Nonintegrable case (stadium billiard).

formulation, being viewed as improperly formed, should be put in a more natural framework that the diffusion coefficient be identified with the inverse of the metric tensor in a Riemannian geometry applied to Brownian motions [16],[17].

We apply the standard theory of Brownian motions in a compact Riemann manifold to the subsystem $\mathcal{R}=\prod(\Omega_{mn}, M_{mn})$ of the level dynamics in the phase space

$$\mathcal{S}\otimes\mathcal{R} = \prod_{n}^{N}(x_n, p_n) \otimes \prod_{m<n}^{Z}(\Omega_{mn}, M_{mn}). \tag{31}$$

Then, the phase space variables for this subsystem (it is possible to choose a canonical set for these) become stochastic variables to describe the heat reservoir against the genuine dynamical subsystem $\mathcal{S}=\prod(x_n, p_n)$ of the total $\mathcal{S}\otimes\mathcal{R}$. The equations of motion for the latter, $\mathcal{S}$, are thus expected to become stochastic differential equations. We outline the result as follows:

1) s. d. e. for Ω_{mn} or for y^i : $\Omega_{mn} = \sum_i A(y)^{mn}_i \circ dy^i$

$$\Omega_{mn} = \frac{1}{x_n - x_m}\circ dB_{mn} \quad \text{or} \quad dy^i = \sum_{(mn)} \frac{A^{-1}(y)^i_{mn}}{x_n - x_m}\circ dB_{mn} \tag{32}$$

The diffusion coefficient matrix for the y-coordinates is given by

$$D_y = A^{-1} g^{-1} A^{t-1} \quad \text{where} \quad g = \mathrm{diag}(\, g_{mn}=(x_m-x_n)^2\,) \ \text{(metric tensor)}$$

Hence, the inverse square-root of the determinant of D_y which plays the role of the stationary distribution for $\mathcal{R}$ is identified as

$$|\det D_y|^{-1/2} = |\det A|\sqrt{g} = |\det A| \prod_{m<n}|x_m - x_n|$$

so that

$$|\det D_y|^{-1/2} dy^1 \cdots dy^Z = \prod_{m<n}|x_m - x_n|\,\Omega_{mn}. \tag{33}$$

(The argument allows a straightforward extension to the case $\nu = 2, 4$.)

2) s. d. e. for x_n and the spacing $x_n - x_m \equiv s_{nm}$

$$dX_n = \frac{1}{\Gamma}(K\,dt + \sigma\, dW_n) \tag{34}$$

$$dS_{nm} = \frac{\beta^{1/2}}{S_{nm}}\circ dB_{mn} + \gamma^{1/2} dW_{mn} \tag{35}$$

derivation The deterministic equations of motion [8]~[11] are:

$$\frac{dx_n}{dt} = p_n , \quad \frac{dp_n}{dt} = 2\sum_{m(\neq n)} \frac{M_{mn}^2}{(x_n - x_m)^3} = 2\sum_{m(\neq n)} (x_n - x_m)\Omega_{t,mn}^2 \qquad (36)$$

The resulting Newtonian equation yields the curvature K [11]. The angular velocity Ω_t is decomposed, by recalling Eq.(32) and $[\Omega_t, H_D] = V$ where V is defined in $H = H^0 + tV$ (17),

such that
$$\Omega_{t,mn} = \langle \Omega_{t,mn} \rangle + \Omega_{fl,mn}$$
$$= \frac{\langle V_{mn} \rangle}{x_n - x_m} + \frac{1}{x_n - x_m} \circ dB_{mn}/dt .$$

By averaging the right hand side of (36),

$$dp_n = K dt + \sum_{m(\neq n)} \left\langle \frac{4\langle V_{mn} \rangle}{x_n - x_m} \right\rangle \circ dB_{mn} = K dt + \sigma dW_n . \qquad (37)$$

These averages, being with distribution over N-1 variables $\{x_n\}$ excepting x_n, yield constant values by virtue of the translational invariance (2). Therefore,

$$K = \sum_{m(\neq n)} \left\langle \frac{2\langle V_{mn} \rangle^2}{x_n - x_m} \right\rangle = 2(N-1)\langle V_{mn}^2 \rangle \left\langle \frac{1}{x_n - x_m} \right\rangle_{m \neq n} \qquad (38a)$$

$$\sigma = 2(N-1)^{1/2} \langle V_{mn} \rangle \left\langle \frac{1}{x_n - x_m} \right\rangle_{m \neq n} . \qquad (38b)$$

To get the spacing distribution for a pair (m,n), the averaging procedure in (36) is modified such that excepting the particular term $2(x_n - x_m)\Omega_{t,mn}^2$ the same average is made to write

$$dp_n = K dt + \frac{4\langle V_{mn} \rangle}{x_n - x_m} \circ dB_{mn} + \sigma dW_n .$$

Similarly,

$$dp_m = K dt + \frac{4\langle V_{mn} \rangle}{x_m - x_n} \circ dB_{mn} + \sigma dW_m \quad (B_{nm} = B_{mn}).$$

Thus, we obtain

$$d(p_n - p_m) = \frac{8\langle V_{mn} \rangle}{x_n - x_m} \circ dB_{mn} + \sigma dW_{nm} \quad (W_{nm} = W_n - W_m).$$

Note that the two Brownian motions B_{mn} and W_{mn} so introduced are stochastically independent of each other:

$$\langle dB_{mn}\ dW_{mn} \rangle = 0. \tag{39}$$

We further modify the deduced s.d.e's for the momentum variables by adding a proper linear-damping term so as to be consistent with the canonical equilibrium $e^{-\beta\mathcal{H}}$ (the so called fluctuation-dissipation theorem)

$$dp_n = (-\Gamma p_n + K)dt + \sigma\, dW_n \tag{40}$$

$$d(p_n - p_m) = -\Gamma(p_n - p_m)dt + \frac{\partial\langle V_{mn}\rangle}{x_n - x_m} \circ dB_{mn} + \sigma\, dW_{mn} \tag{41}$$

with the damping constant related to σ through $\Gamma = (2\beta)^{-1}\sigma^2$. Stochastic diff. eqs.(34) and (35) are the consequence of the dynamical coarse-graining $\Gamma t \gg 1$.

remark Yukawa's distribution function (24) can be regarded as the stationary distribution for the process (35) of all the spacings, provided 2Z Brownian motions $\{B_{mn}\}$ and $\{W_{mn}\}$ are mutually independent. Obviously, this is not true in the above argument. If, however, each W_{mn} arises from external origins, the independence could be recovered.

4.3 the maximum information principle for Yukawa's distribution

A Gaussian distribution can be characterized by its entropy that, of all possible probability distributions represented in the Euclidian space with densities having a fixed mean vector and a fixed variance matrix , it has the maximum of entropy, $H[p] = -\int p(x)\log p(x)dx$, [18]. This fact can be used to make a similar characterization of Yukawa's distribution (24). This is meaningful, since as already discussed in Sec. 3 the canonical distribution $e^{-\beta\mathcal{H}-\gamma Q}$ from which the expression (24) is deduced by partial integrations must have no obvious reason by itself because of the complete integrability of our level dynamics. We can state as follows.

Let us modify the decomposition of the total system into the system $\mathcal{S}$ and the reservoir $\mathcal{R}$, (31), such that

$$\mathcal{S} = \{x_n\}, \quad \mathcal{K} = \{p_n\} \otimes \prod_{m<n}^{Z} (\Omega_{mn}, M_{mn}). \tag{42}$$

Then, of all possible canonical distributions of the form $\frac{1}{Z} e^{-\sum \beta_\nu Q_\nu}$ ($\{Q_\nu\}$ is

the set of all the constants of the level motion) defined in the phase space $\mathcal{S} \otimes \mathcal{R}$ with only the two fixed variances (besides <p>=<M>=0)

$$\langle p_n^2 \rangle = \beta^{-1} \quad (=1 \text{ in Eq. (24)}) \quad \text{and} \quad \langle M_{mn}^2 \rangle = \gamma^{-1},$$

the specific form $\frac{1}{Z} e^{-\beta \mathcal{H} - \gamma Q}$ has the maximum of conditional entropy

$$H[p|\{x_n\}] = -\int p(y|x) \log p(y|x)\, dy \equiv h(x), \quad dy = \prod dp_n \prod dM_{mn}$$

and in terms of its maximum value, $h_m(x)$, $P(\{x_n\}, \gamma) = C' \exp(-h_m(x))$.

ACKNOWLEDGMENTS

We thank Profs. I. Ojima and K. Nakamura for useful discussions of the level dynamics. We also thank Prof. E. Heller for stimulating comments on the stadium billiard problem.

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Appendix A

Eigenvalues of a Circular Billiard

Every eigenvalues of a circular billiard (the boundary-value problem for a two dimmentional wave equation with the Dirichlet condition, $\psi = 0$ on a circle) can be calculated from the zeros of the Bessel function. The stationary Shrödinger equation, written in the polar coordinate, is (in the unit of $\hbar^2/2m = 1$)

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2})\psi(r,\theta) + E\psi(r,\theta) = 0. \qquad \text{(A1)}$$

An eigen function $\psi(r,\theta)$ of (A1) is represented by the Bessel function in the following form

$$\psi(r,\theta) = c\ J_{2n}(\frac{r}{R_0} j_{2n,k})\ \sin 2n\theta, \qquad (\text{for} \quad n = 1,2,\ldots) \tag{A2}$$

where $j_{2n,k}$ means the k-th zero of the 2n-th Bessel function, R_0 is the radius of the circle, and c is the normalization constant. (We concentrate here on the anti-symmetric states to x and y axis.) It is easy to verify that the wave function (A2) satisfies (A1), and then

$$E = (\frac{j_{2n,k}}{R_0})^2. \tag{A3}$$

The calculation of the zeros $j_{n,k}$ is easy, because the zeros of $(n+1)$-th Bessel function is always sandwiched between two adjacent zeros of n-th Bessel function, such that

$$j_{n,k} < j_{n+1,k} < j_{n,k+1}. \tag{A4}$$

Therefore, if we know the zeros of n-th Bessel function, we can get $(n+1)$-th one by the bisection method. All the eigenvalues necessary for plotting Fig.2(a) have been obtained by applying this method.

Appendix B

Eigenvalues of a Stadium Billiard

The standard numerical method to calculate eigenvalues of the nonintegrable billiard system is the Green's function method[20], which is known as Boundary Element method (BEM) in the field of engineering. But we use another method (the plane wave method[21]) here to calculate the eigenvalues of a stadium billiard.

Eigen equation of a billiard system is

$$\begin{aligned} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2})\psi(x,y) + E\psi(x,y) = 0 \qquad & (\text{inside of the boundary}) \\ \psi(x,y) = 0 \qquad & (\text{outside of the boundary}). \end{aligned} \tag{B1}$$

In many cases, the eigen-functions for the anti-symmetric subspace are represented in the following form,

$$\psi_n(x,y) = c \int_0^{\pi/2} d\theta\ \phi_n(\theta)\ \sin(xk\cos\theta)\ \sin(yk\sin\theta), \tag{B2}$$

where $k^2 = E$. To calculate eigenvalues, we have only to search for the values of the wave number k and the coefficients $\phi_n(\theta)$ so that $\psi_n(x,y) = 0$ on the boundary. This is done numerically by calculating the zeros of the determinant of the following matrix,

$$\mathbf{A} = (a_{ij}) \qquad a_{ij} = \sin(kx_i\cos\theta_j)\cos(ky_i\sin\theta_j), \tag{B3}$$

where (x_i, y_i) is the points on the boundary and $\theta_j = (2j+1)\pi/4N$ for $i,j = 0,1,\ldots,N-1$.

Although this method is not supported mathematically, we know that it works well for a stadium billiard from the experience. If we apply this method to other systems, it is necessary to check by Weyl's formula that no eigenvalue is missing.